THEOREM: THE CONSTANT RULE

Let k be a real number.

$$\int k\,dx = kx + C$$

Example 1: Find the indefinite integral.

$$\int -3\,dx = \boxed{-3x + C}$$

THEOREM: THE POWER RULE

Let n be a rational number.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

Example 2: Find the following indefinite integrals.

a. $\int x^{-5}\,dx = \dfrac{x^{-4}}{-4} + C$

$$= -\frac{1}{4}x^{-4} + C$$

OR

$$= -\frac{1}{4x^4} + C$$

OR

$$= -\frac{x^{-4}}{4} + C$$

b. $\int x^{1/2}dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$

$$= \boxed{\frac{2}{3}x^{3/2} + C}$$

THEOREM: THE CONSTANT MULTIPLE RULE

If f is an integrable function and c is a real number, then cf is also integrable and

$$\int cf(x)\,dx = c\int f(x)dx$$

Example 3: Find the area of the region bounded by $f(x) = 2x^3$, $x = 1$, $x = 3$, and $y = 0$.

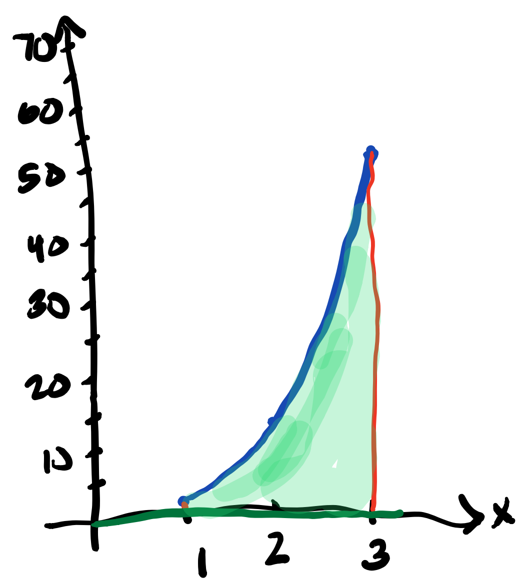
$$A = \int_{x=1}^{x=3} 2x^3\,dx$$

$$A = \frac{x^4}{2}\Bigg|_{x=1}^{x=3}$$

$$A = \frac{1}{2}(3^4 - 1^4)$$

$$A = \frac{1}{2}\cdot 80$$

$$\boxed{A = 40 \text{ sq. units}}$$



THEOREM: THE SUM AND DIFFERENCE RULES

The sum (or difference) of two integrable functions f and g is itself integrable. Moreover, the antiderivative of $f+g$ (or $f-g$) is the sum (or difference) of the antiderivatives of f and g.

$$\int\left[f(x)+g(x)\right]dx=\int f(x)\,dx+\int g(x)\,dx$$

$$\int\left[f(x)-g(x)\right]dx=\int f(x)\,dx-\int g(x)\,dx$$

Example 4: Find the indefinite integral.

a. $\int\left(\frac{\sqrt{x}-5x^2}{\sqrt{x}}\right)dx$

$=\int(x^{1/2}-5x^2)x^{-1/2}\,dx$

$=\int(1-5x^{3/2})\,dx$

$=x-\frac{5x^{5/2}}{5/2}+C$

$=\boxed{x-2x^{5/2}+C}$

Evil plan:
using algebra to "break it up"

b. $\int(x^3+1)^2\,dx$

$=\int(u)^2\left(\frac{du}{3x^2}\right)$

CRAP!

$\int(x^3+1)^2\,dx$

$=\int(x^6+2x^3+1)\,dx$

$=\boxed{\frac{1}{7}x^7+\frac{1}{2}x^4+x+C}$

Evil plan:

~~u-sub~~

~~$u=x^3+1$~~

~~$\frac{du}{dx}=3x^2$~~

~~$dx=\frac{du}{3x^2}$~~

Algebra
- expanding the binomial sum

c. $\displaystyle\int_3^5 \frac{5+6x+x^2}{5+x}\,dx$

Evil plan:

Use algebra to rewrite
- factor or
- poly. long division

$$\begin{array}{r} x+1 \\ (x+5)\overline{)x^2+6x+5} \\ -(x^2+5x) \\ \hline x+5 \\ -(x+5) \\ \hline 0 \end{array}$$

$$= \int_{x=3}^{x=5} (x+1)\,dx$$

$$= \frac{1}{2}x^2 + x \Big|_{x=3}^{x=5}$$

$$= \left[\left(\tfrac{1}{2}(5)^2 + (5)\right) - \left(\tfrac{1}{2}(3)^2 + (3)\right)\right]$$

$$= \frac{25}{2} + 5 - \frac{9}{2} - 3$$

$$= 8 + 2$$

$$= \boxed{10}$$

THEOREM: ANTIDERIVATIVES OF THE TRIGONOMETRIC FUNCTIONS

$\int \sin x dx = -\cos x + C$	$\int \cos x dx = \sin x + C$
$\int \csc x \cot x dx = -\csc x + C$	$\int \sec x \tan x dx = \sec x + C$
$\int \sec^2 x dx = \tan x + C$	$\int \csc^2 x dx = -\cot x + C$
$\int \tan x dx = -\ln\lvert\cos x\rvert + C$	$\int \cot x dx = \ln\lvert\sin x\rvert + C$
$\int \sec x dx = \ln\lvert\sec x + \tan x\rvert + C$	$\int \csc x dx = -\ln\lvert\csc x + \cot x\rvert + C$

Example 5: Integrate.

a. $\int \sec^2 x dx = \boxed{\tan x + C}$

b. $\int(-\csc\theta+\csc\theta\cot\theta)\,d\theta$

$= -(-\ln|\csc\theta+\cot\theta|) + (-\csc\theta) + C$

$= \boxed{\ln|\csc\theta+\cot\theta| - \csc\theta + C}$

c. $\int 3\tan x\,dx = -3\ln|\cos x| + C$

$= \ln|(\cos x)^{-3}| + C$

$= \boxed{\ln|\sec^3 x| + C}$

d. $\int \frac{1}{1+\cos\theta}\,d\theta$

$\int \frac{1}{(1+\cos\theta)}\cdot\frac{(1-\cos\theta)}{(1-\cos\theta)}\,d\theta$

$= \int \frac{1-\cos\theta}{1-\cos^2\theta}\,d\theta$

$= \int \frac{1-\cos\theta}{\sin^2\theta}\,d\theta$

$= \int\left(\frac{1}{\sin^2\theta} - \frac{\cos\theta}{\sin^2\theta}\right)d\theta$

$= \int\left(\csc^2\theta - \frac{1}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta}\right)d\theta$

$= \int(\csc^2\theta - \csc\theta\cot\theta)\,d\theta$

$\Rightarrow = -\cot\theta - (-\csc\theta) + C$

$= \boxed{-\cot\theta + \csc\theta + C}$

Evil plan:

$\cancel{\frac{1}{1+2} \overset{?}{=} \frac{1}{1} + \frac{1}{2}}$

$\cancel{\frac{1}{3} \neq \frac{3}{2}}$

$\cancel{u = 1+\cos\theta}$

$\cancel{\frac{du}{dx} = -\sin\theta}$

trig identity

Pyth. conjugate

THEOREM: ANTIDIFFERENTIATION OF A COMPOSITE FUNCTION

Let g be a function whose range is an interval I and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then

sine function

cosine function

$$\int f\left(\overbrace{g(x)}^{6x}\right)\overbrace{g'(x)}^{6}\,dx = F\left(\overbrace{g(x)}^{6x}\right)+C$$

Letting $u = g(x)$ gives $du = g'(x)\,dx$ and

$$\int f(u)\,du = F(u)+C$$

Example 6: Find the following definite and indefinite integrals.

a. $\int \left(x\sqrt{1-x}\right)dx$

$= \int x(u)^{1/2}(-du)$

$= -\int (1-u)u^{1/2}$

$= -\int (u^{1/2} - u^{3/2})du$

$= -\frac{u^{3/2}}{3/2} + \frac{u^{5/2}}{5/2} + C$

$= \boxed{-\frac{2}{3}(1-x)^{3/2} + \frac{2}{5}(1-x)^{5/2} + C}$

Evil plan

1) $u = 1-x$

$\frac{du}{dx} = -1$

$dx = -du$

2) $x = 1-u$

3) use algebra

$$\int f[g(x)]g'(x)\,dx = F(g(x)) + C$$

pattern recognition

b. $\int x(5-2x^2)^5\,dx$

$= \frac{1}{-4}\int (5-2x^2)^5(-4x)\,dx$

$= -\frac{1}{4}\left[\frac{(5-2x^2)^6}{6}\right] + C$

$= \boxed{-\frac{1}{24}(5-2x^2)^6 + C}$

$\int \cancel{x}(u)^5\left(\frac{du}{-4\cancel{x}}\right)$

$= -\frac{1}{4}\int u^5\,du$

$= -\frac{1}{4}\frac{u^6}{6} + C$

$= \boxed{\frac{-1}{24}(5-2x^2)^6 + C}$

Evil plan:

$g(x) = 5-2x^2$

$g'(x) = -4x$

$f(\ ) = (\ )^5$

$F(\ ) = \frac{(\ )^6}{6}$

$u = 5-2x^2$

$\frac{du}{dx} = -4x$

$dx = \frac{du}{-4x}$

c. $\int \cos^2 3x\,dx$

$= \int (\cos 3x)^2\,dx$

$= \int \frac{1+\cos[2\cdot 3x]}{2}\,dx$

$= \frac{1}{2}\int (1+\cos 6x)\,dx$

$= \frac{1}{2}\int \left(1+\frac{6}{6}\cos 6x\right)dx$

$= \frac{1}{2}\left[\int 1\,dx + \frac{1}{6}\int \cos 6x\,(6\,dx)\right]$

$= \frac{1}{2}\left[x + \frac{1}{6}\sin 6x\right] + C$

$= \frac{1}{2}x + \frac{1}{12}\sin 6x + C$

Evil Plan

~~u-sub~~

~~$u = \cos 3x$~~

~~$\frac{du}{dx} = -3\sin 3x$~~

Trig identity

recall that

$\cos 2A = 2\cos^2 A - 1$

$1 + \cos 2A = 2\cos^2 A$

$\to \cos^2 A = \frac{1+\cos 2A}{2}$

$\to \sin^2 A = \frac{1-\cos 2A}{2}$

hmm...

$u = 6x$

$\frac{du}{dx} = 6 \to dx = \frac{du}{6}$

d. $\displaystyle\int_{x=\pi/4}^{x=\pi/3} \tan^3 x \sec^2 x\,dx$

$$= \int_{u=1}^{u=\sqrt{3}} (u)^3 \cancel{\sec^2 x}\left(\frac{du}{\cancel{\sec^2 x}}\right)$$

$$= \int_{u=1}^{u=\sqrt{3}} u^3\,du$$

$$= \frac{1}{4}u^4 \Big|_{u=1}^{u=\sqrt{3}}$$

$$= \frac{1}{4}\left((\sqrt{3})^4 - (1)^4\right)$$

$$= \frac{1}{4}\left(3^2 - 1\right)$$

$$= \frac{1}{4}(8)$$

$$= \boxed{2}$$

Evil plan

$u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$dx = \frac{du}{\sec^2 x}$

$u(x) = \tan x$

$u(\pi/4) = \tan\frac{\pi}{4} = 1$

$u(\pi/3) = \tan\frac{\pi}{3} = \sqrt{3}$

natural logarithms

Theorem: LOG RULE FOR INTEGRATION

Let u be a differentiable function of x.

1. $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$
2. $\displaystyle\int \frac{1}{u}\,du = \ln|u| + C$

Theorem: INTEGRATION RULES FOR EXPONENTIAL FUNCTIONS

Let u be a differentiable function of x.

natural exp.

1. $\displaystyle\int e^x\,dx = e^x + C$
2. $\displaystyle\int e^u\,du = e^u + C$
3. $\displaystyle\int a^x\,dx = \left(\frac{1}{\ln a}\right)a^x + C$, a is a positive real number, $a \neq 1$

base other than e

Example 7: Find the following indefinite integrals and evaluate the definite integrals.

a. $\int \frac{5t^2 - t - 1}{2 - t}\, dt$

$= \int \left(-5t - 9 + \frac{17}{2-t}\right) dt$

$= \int (-5t - 9)\, dt + 17 \int \frac{1}{2-t}\, dt$

$= -\frac{5t^2}{2} - 9t - 17 \ln|2-t| + C$

$= \boxed{-\frac{5}{2}t^2 - 9t - \ln\left|(2-t)^{17}\right| + C}$

Evil plan

use algebra

↳ long division

$$\begin{array}{r} -5t - 9 + \frac{17}{2-t} \\ (-t+2)\overline{)\,5t^2 - t - 1} \\ -(5t^2 - 10t) \downarrow \\ \hline 9t - 1 \\ -(9t - 18) \\ \hline 17 \end{array}$$

$u = 2 - t$

$\frac{du}{dt} = -1$

b. $\int \frac{5}{\left(\sqrt{x}\ln x\right)^2}\, dx$

$= 5\int \frac{1}{x(\ln x)^2}\, dx$

$= 5\int x^{-1} (\ln x)^{-2}\, dx$

$= 5\int \cancel{x^{-1}}\, (u)^{-2}\, (\cancel{x}\, du)$

$= 5\int u^{-2}\, du$

$= 5\frac{u^{-1}}{-1} + C$

$= \boxed{-5(\ln x)^{-1} + C \rightarrow = -\frac{5}{\ln x} + C}$

$(\sin x)^{-1} = \frac{1}{\sin x}$

$(\ln x)^{-1} \neq \ln(\ln x)$

Evil plan

algebra

↳ multiply

$(\sqrt{x}\ln x)^2 = (\sqrt{x}\ln x)(\sqrt{x}\ln x)$

$= x(\ln x)^2$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$dx = x\, du$

consider $(\ln x)^2$ vs $\ln x^2$

$(\ln x)(\ln x)$ vs $\ln(x \cdot x)$

$2\ln x$

c. $\int \frac{1}{x^{2/3}\left(1+x^{1/3}\right)}dx$

$= \int \frac{1}{\cancel{x^{2/3}}(u)} \cancel{(3x^{2/3})}du$

$= 3\int \frac{du}{u}$

$= 3\ln|u| + C$

$= 3\ln|1+x^{1/3}| + C$

$= \boxed{\ln|(1+x^{1/3})^3| + C}$

Evil plan

$u = 1+x^{1/3}$

$\frac{du}{dx} = \frac{1}{3}x^{-2/3}$

$dx = 3x^{2/3}du$

d. $\int_1^2 \frac{e^x+e^{-x}}{e^x-e^{-x}}dx$

$u=e^2-e^{-2}$

$= \int_{u=e-e^{-1}}^{u=e^2-e^{-2}} \frac{\cancel{e^x+e^{-x}}}{u} \cdot \frac{du}{\cancel{e^x+e^{-x}}}$

$= \int_{u=e-e^{-1}}^{u=e^2-e^{-2}} \frac{du}{u}$

$= \ln|u| \Big|_{u=e-e^{-1}}^{u=e^2-e^{-2}}$

$= \ln|(e^2-e^{-2})| - \ln|(e-e^{-1})|$

Evil plan

$u = e^x - e^{-x}$

$\frac{du}{dx} = e^x - (-e^{-x})$

$dx = \frac{du}{e^x+e^{-x}}$

$u(x) = e^x - e^{-x}$

$u(2) = e^2 - e^{-2}$

$u(1) = e - e^{-1}$

$$= \ln\left|\frac{\left(e^2 - \frac{1}{e^2}\right)}{\left(e - \frac{1}{e}\right)} \cdot \frac{e^2}{e^2}\right|$$

$$= \ln\left|\frac{e^4 - 1}{e^3 - e}\right|$$

$$= \ln\left|\frac{e^4 - 1}{e(e^2-1)}\right|$$

$$= \ln\left|\frac{\cancel{(e^2-1)}(e^2+1)}{e\cancel{(e^2-1)}}\right|$$

$$= \ln(e^2+1) - \ln e$$

$$= \boxed{-1 + \ln(e^2+1)}$$

Review for 8.3

Memorize Pythagorean Id.

1) $\sin^2\theta + \cos^2\theta = 1$

2) $1 + \tan^2\theta = \sec^2\theta$

3) $1 + \cot^2\theta = \csc^2\theta$

and . . .

Double angle

1) $\sin 2\theta = 2\sin\theta\cos\theta$

2) $\cos 2\theta = \cos^2\theta - \sin^2\theta$
$= 2\cos^2\theta - 1$
$= 1 - 2\sin^2\theta$

Power reducing

1) $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

2) $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

e. $\int_{\pi/6}^{\pi/4} \sec^2 x dx$

f. $\int_{-\pi/2}^{\pi/2} \sin x \cos^2 x dx$

Theorem: INTEGRALS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

Let u be a differentiable function of x, and let $a > 0$.

$$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C \qquad \int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + C \qquad \int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$$

$\sin^{-1}\frac{u}{a}$ etc . . .

Example 7: Find the following indefinite integrals and evaluate the definite integrals.

a. $\displaystyle\int \frac{t}{t^4+2}\,dt$

$$= \int \frac{\cancel{t}}{u^2+(\sqrt{2})^2} \cdot \frac{du}{2\cancel{t}}$$

$$= \frac{1}{2}\int \frac{du}{(\sqrt{2})^2+u^2}$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{2}} \arctan\frac{u}{\sqrt{2}} + C$$

$$= \boxed{\frac{\sqrt{2}}{4}\arctan\left(\frac{t^2}{\sqrt{2}}\right) + C}$$

Evil plan

$\cancel{u = t^4 + 2}$

$\cancel{\frac{du}{dt} = 4t^3}$

inverse trig?

arctan( )?

$u = t^2$

$\frac{du}{dt} = 2t$

$dt = \frac{du}{2t}$

$a = \sqrt{2}$

b. $\int \frac{dx}{\sqrt{5-4x-x^2}}$

$$= \int \frac{dx}{\sqrt{(3)^2-(x+2)^2}}$$

$$= \boxed{\arcsin\left(\frac{x+2}{3}\right)+C}$$

Evil plan

inv. trig.

$\hookrightarrow$ complete square

$$-x^2-4x+5 = -(x^2+4x+(2)^2)+5+4$$

$$= 9-(x+2)^2$$

$a = 3$

$u = x+2$

$\frac{du}{dx} = 1$

c. $\int \frac{dx}{\sqrt{e^{2x}-25}}$

$$= \int \frac{dx}{\sqrt{(e^x)^2-(5)^2}}$$

$$= \int \frac{du/u}{\sqrt{u^2-5^2}}$$

$$= \int \frac{du}{u\sqrt{u^2-5^2}}$$

$$= \frac{1}{5}\operatorname{arcsec}\frac{|u|}{5}+C$$

$$= \boxed{\frac{1}{5}\operatorname{arcsec}\left(\frac{e^x}{5}\right)+C}$$

evil plan

remember $e^{2x} = (e^x)^2$

$u = e^x$

$\frac{du}{dx} = e^x$

$dx = \frac{du}{e^x}$

$dx = \frac{du}{u}$